

Power solution expansions of the analogue to the first Painleve equation

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Abstract

The fourth-order analog to the first Painlevé equation is studied. All power expansions for solutions of this equation near points $z = 0$ and $z = \infty$ are found. The exponential additions to the expansion of solution near $z = \infty$ are computed. The obtained results confirm the hypothesis that the fourth-order analog of the first Painlevé equation determines new transcendental functions. By means of the methods of power geometry the basis of the plane lattice is also calculated.

1. Introduction.

In [1] the hierarchy of the first Painlevé equation was suggested. It can be described by the relation

$$L^n[w] = z \tag{1.1}$$

where L^n is the Lenard's operator, which is determined by the relation [2]

$$\frac{d}{dz}L^{n+1} = L_{zzz}^n - 4wL_z^n - 2w_zL^n, \quad L^0[w] = -\frac{1}{2} \tag{1.2}$$

Assumed that $n = 0$ in (1.2), we have $L^1[w] = w$. In case of $n = 2$ using (1.1) we get the first Painlevé equation [3]

$$w_{zz} - 3w^2 = z \quad (1.3)$$

If $n = 3$ in (1.1), we get the fourth-order equation [1, 2]

$$w_{zzzz} - 10w w_{zz} - 5w_z^2 + 10w^3 = z \quad (1.4)$$

Using $n = 4$, we obtain the sixth-order equation from (1.1)

$$w_{zzzzzz} - 14w w_{zzzz} - 28 w_z w_{zzz} - 21 w_{zz}^2 + 70 w w_z^2 - 35 w^4 = z \quad (1.5)$$

Equation (1.4) is used in describing of the waves on water [4, 5] and in the Henon-Heiles model, which characterizes the behavior of star in the middle field of galaxy [6–8].

In papers [9–23] it was shown that equation (1.4) has properties, that are typical for the Painlevé equations $P_1 \div P_6$. Equation (1.4) belongs to the class of exactly solvable equations, as it has Lax pair and a lot of other typical properties of the exactly solvable equations. However it doesn't have the first integrals in the polynomial form, that is one of the features of the Painlevé equations. Equation (1.4) seems to determine new transcendental functions just as equations $P_1 \div P_6$, although the rigorous proof of the irreducibility of equation (1.4) is now the open problem.

Thereupon the study of all the asymptotic forms and power expansions of equation (1.4) is the important stage of the analysis of this equation, as this fact indirectly confirms the irreducibility of equation (1.4).

Let's find all the power expansions for the solution of equation (1.4) in the form of

$$w(z) = c_r z^r + \sum_s c_s z^s \quad (1.6)$$

at $z \rightarrow 0$, then $\omega = -1$, $s > r$ and at $z \rightarrow \infty$, then $\omega = 1$, $s < r$.

For that we use the methods of power geometry [24, 25] by analogy with [26].

2. The general properties of equation (1.4).

Let's consider the fourth-order equation (1.4)

$$f(z, w) \stackrel{\text{def}}{=} w_{zzzz} - 10ww_{zz} - 5w_z^2 + 10w^3 - z = 0 \quad (2.1)$$

For monomials of equation (2.1) we have points $M_1 = (-4, 1)$, $M_2 = (-2, 2)$, $M_3 = (-2, 2)$, $M_4 = (0, 3)$, $M_5 = (1, 0)$.

The carrier of equation is defined by four points $Q_1 = M_1$, $Q_2 = M_4$, $Q_3 = M_5$ and $Q_4 = M_2 = M_3$. Their convex hull Γ is the triangle (fig. 1).

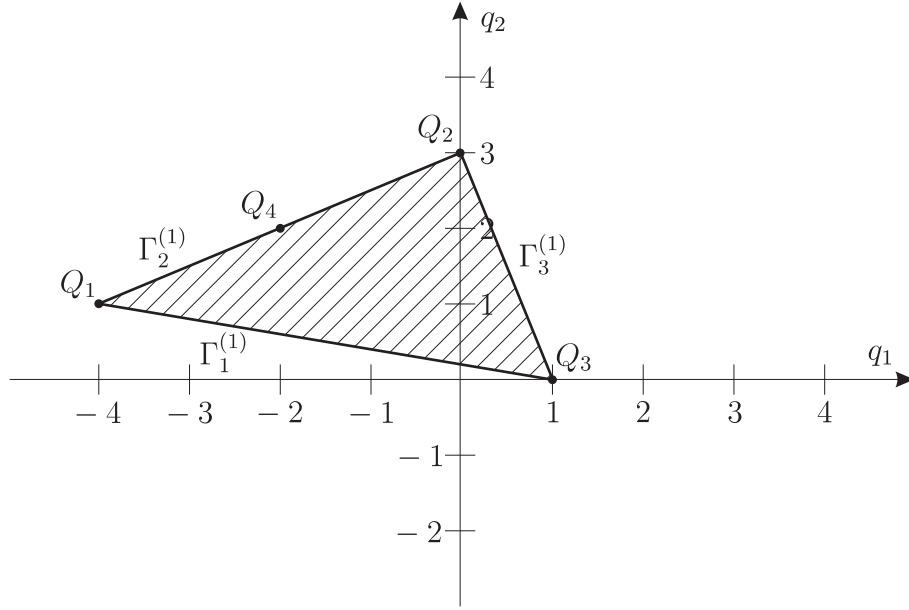


Figure 1:

This triangle has apexes Q_j ($j = 1, 2, 3$) and edges $\Gamma_1^{(1)} = [Q_3, Q_1]$, $\Gamma_2^{(1)} = [Q_1, Q_2]$, $\Gamma_3^{(1)} = [Q_2, Q_3]$

Outward normal vectors N_j ($j = 1, 2, 3$) of edges $\Gamma_j^{(1)}$ ($j = 1, 2, 3$) are determined by vectors

$$N_1 = (-1, -5), \quad N_2 = (-1, 2), \quad N_3 = (3, 1) \quad (2.2)$$

The normal cones $U_j^{(1)}$ to edges $\Gamma_j^{(1)}$ are

$$U_j^{(1)} = \mu N_j, \quad \mu > 0, \quad j = 1, 2, 3 \quad (2.3)$$

They and the normal cones $U_j^{(0)}$ of apexes $\Gamma_j^{(0)} = Q_j$ ($j = 1, 2, 3$) are represented at fig. 2.

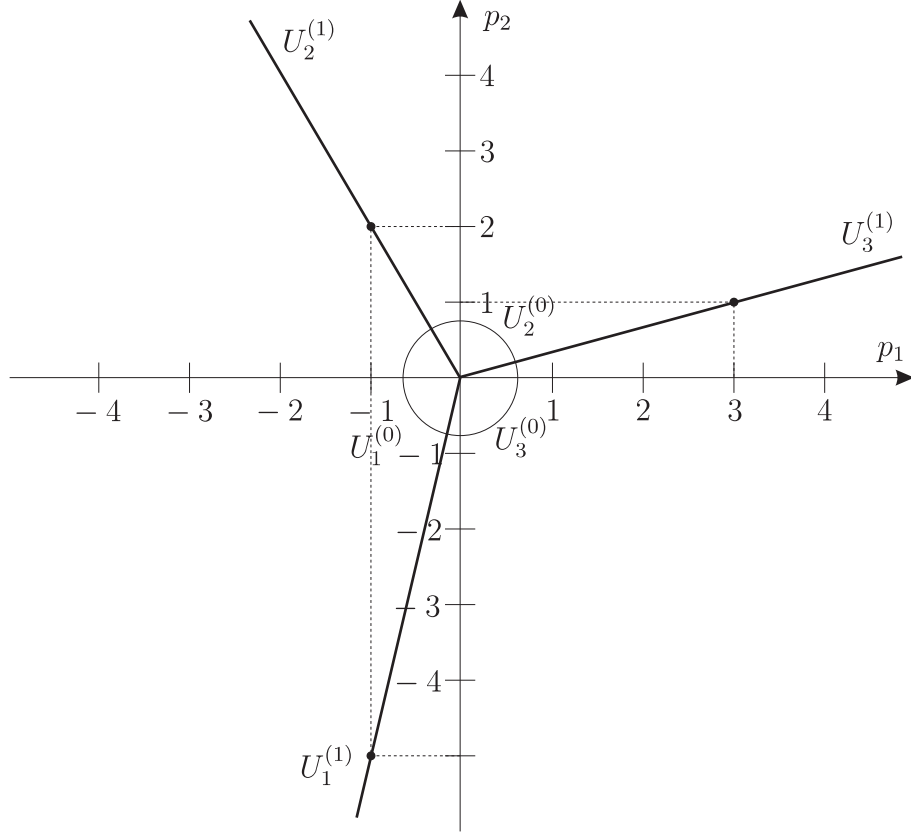


Figure 2:

If the carrier of equation (2.1) is moved by vector $-Q_3$, then it is situated at the lattice \mathbf{Z} , formed by vectors

$$Q_1 - Q_3 = (-5, 1), \quad Q_2 - Q_1 = (-1, 3) \quad (2.4)$$

We choose the basis of the lattice as

$$B_1 = (-5, 1), \quad B_2 = (-3, 2) \quad (2.5)$$

Let's study solutions, corresponding to the bounds $\Gamma_j^{(d)}$, $d = 0, 1$; $j = 1, 2, 3$ in view of the reduced equations, conforming to apexes $\Gamma_j^{(0)}$ ($j = 1, 2, 3$)

$$\hat{f}_1^{(0)} \stackrel{\text{def}}{=} w_{zzzz} = 0 \quad (2.6)$$

$$\hat{f}_2^{(0)} \stackrel{def}{=} 10 w^3 = 0 \quad (2.7)$$

$$\hat{f}_3^{(0)} \stackrel{def}{=} -z = 0 \quad (2.8)$$

and reduced equations, conforming to edges $\Gamma_j^{(1)} (j = 1, 2, 3)$

$$\hat{f}_1^{(1)} \stackrel{def}{=} w_{zzzz} - z = 0 \quad (2.9)$$

$$\hat{f}_2^{(1)} \stackrel{def}{=} w_{zzzz} - 10 w w_{zz} - 5 w_z^2 + 10 w^3 = 0 \quad (2.10)$$

$$\hat{f}_3^{(1)} \stackrel{def}{=} 10 w^3 - z = 0 \quad (2.11)$$

Note, that the reduced equations (2.7) and (2.8) are the algebraic ones. According to [25] they don't have non-trivial power or non-power solutions.

3. Solutions, corresponding to apex Q_1 .

Apex $Q_1 = (-4, 1)$ is corresponded to reduced equation (2.6).

Let's find the reduced solutions

$$w = c_r z^r, \quad c_r \neq 0 \quad (3.1)$$

for $\omega(1, r) \in U_1^{(0)}$.

Since $p_1 < 0$ in the cone $U_1^{(0)}$, then $\omega = -1$, $z \rightarrow 0$ and the expansions are the ascending power series of z . The dimension of the bound $d = 0$, therefor

$$g(z, w) = w^4 w^{-1} w_{zzzz} \quad (3.2)$$

We get the characteristic polynomial

$$\chi(r) \stackrel{def}{=} g(z, z^r) = r(r-1)(r-2)(r-3) \quad (3.3)$$

Its roots are

$$r_1 = 0, \quad r_2 = 1, \quad r_3 = 2, \quad r_4 = 3 \quad (3.4)$$

Let's explore all these roots.

The root $r_1 = 0$ is corresponded to vector $R = (1, 0)$ and vector $\omega R \in U_1^{(0)}$.

We obtain the family $\mathcal{F}_1^{(1)} 1$ of reduced solutions $y = c_0$, where $c_0 \neq 0$ is arbitrary constant and $\omega = -1$. The first variation of equation (2.6)

$$\frac{\delta \hat{f}_1^{(0)}}{\delta w} = \frac{d^4}{dz^4} \quad (3.5)$$

gives operator

$$\mathcal{L}(z) = \frac{d^4}{dz^4} \neq 0 \quad (3.6)$$

Its characteristic polynomial is

$$\nu(k) = z^{4-k} \mathcal{L}(z) z^k = k(k-1)(k-2)(k-3) \quad (3.7)$$

Equation

$$\nu(k) = 0 \quad (3.8)$$

has four roots

$$k_1 = 0, \quad k_2 = 1, \quad k_3 = 2, \quad k_4 = 3 \quad (3.9)$$

As long as $\omega = -1$ and $r = 0$, then the cone of the problem is

$$\mathcal{K} = \{k > 0\} \quad (3.10)$$

It contains the critical numbers $k_2 = 1$, $k_3 = 2$ and $k_3 = 3$. Expansions for the solutions, corresponding to reduced solution (3.1) can be presented in the form

$$w = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \sum_{k=4}^{\infty} c_k z^k \quad (3.11)$$

where all the coefficients are constants, $c_0 \neq 0$, c_1 , c_2 , c_3 are arbitrary ones and c_k ($k \geq 4$) are uniquely defined. Denote this family as $G_1^{(0)} 1$. Expansion (3.11) with taking into account eight terms is

$$\begin{aligned} w(z) = & c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \left(\frac{5}{6} c_0 c_2 - \frac{5}{12} c_0^3 + \frac{5}{24} c_1^2 \right) z^4 + \\ & + \left(\frac{1}{120} \alpha - \frac{1}{4} c_1 c_0^2 + \frac{1}{2} c_0 c_3 + \frac{1}{3} c_1 c_2 \right) z^5 + \\ & + \left(\frac{7}{36} c_2 c_0^2 - \frac{5}{36} c_0^4 - \frac{1}{72} c_0 c_1^2 + \frac{1}{9} c_2^2 + \frac{1}{4} c_1 c_3 \right) z^6 + \\ & + \left(\frac{1}{36} c_1^3 + \frac{1}{504} c_0 \alpha + \frac{1}{12} c_3 c_0^2 + \frac{1}{6} c_0 c_1 c_2 - \frac{5}{36} c_1 c_0^3 + \frac{1}{6} c_2 c_3 \right) z^7 + \dots \end{aligned}$$

Let's explore root $r_2 = 1$. The cone of the problem is $\mathcal{K} = \{k > 1\}$. It contains the critical numbers $k_2 = 2$, $k_3 = 3$. The expansion of solution, corresponding to the reduced solution

$$\mathcal{F}_1^{(1)}2: w = c_1 z$$

can be written as

$$w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \sum_{k=4}^{\infty} c_k z^k \quad (3.12)$$

where $c_1 \neq 0$, c_2 and c_3 are the arbitrary constants. Denote this family as $G_1^{(0)}2$. The expansion of solutions (3.12) with taking into account seven terms is

$$\begin{aligned} w(z) = & c_1 z + c_2 z^2 + c_3 z^3 + \frac{5}{24} c_1^2 z^4 + \left(\frac{1}{120} + \frac{1}{3} c_1 c_2 \right) z^5 + \\ & + \left(\frac{1}{4} c_1 c_3 + \frac{1}{9} c_2^2 \right) z^6 + \left(\frac{1}{6} c_2 c_3 + \frac{1}{36} c_1^3 \right) z^7 + \dots \end{aligned}$$

For root $r_2 = 2$ the cone of the problem is $\mathcal{K} = \{k > 2\}$. The critical number is $k_3 = 3$. The expansion of the solutions, corresponding to the reduced solution

$$\mathcal{F}_1^{(1)}3: w = c_2 z^2$$

takes the form

$$w = c_2 z^2 + c_3 z^3 + \sum_{k=4}^{\infty} c_k z^k \quad (3.13)$$

Denote this family as $G_1^{(0)}3$. Expansion (3.13) with taking into account eight terms is

$$\begin{aligned} w(z) = & c_2 z^2 + c_3 z^3 + \frac{1}{120} z^5 + \frac{1}{9} c_2^2 z^6 + \frac{1}{6} c_2 c_3 z^7 + \frac{1}{16} c_3^2 z^8 + \\ & + \frac{1}{1134} c_2 z^9 + \left(\frac{5}{648} c_2^3 + \frac{41}{60480} c_3 \right) z^{10} + \dots \end{aligned} \quad (3.14)$$

For root $r_3 = 3$ the cone of the problem is $\mathcal{K} = \{k > 3\}$. There is no critical number here. The expansion of solutions, corresponding to the reduced solution

$$\mathcal{F}_1^{(1)}4: w = c_3 z^3$$

takes the form

$$w(z) = c_3 z^3 + \sum_{k=4}^{\infty} c_k z^k \quad (3.15)$$

Denote this family as $G_1^{(0)}4$. The expansion (3.15) with taking into account four terms is

$$w(z) = c_3 z^3 + \frac{1}{120} z^5 + \frac{1}{16} c_3^2 z^8 + \frac{41}{60480} c_3 z^{10} + \dots \quad (3.16)$$

The expansions of solutions converge for sufficiently small $|z|$. The existence and analyticity of expansions (3.11), (3.12), (3.13) and (3.15) follow from Cauchy theorem.

4. Solutions, corresponding to edge $\Gamma_1^{(1)}$.

Edge $\Gamma_1^{(1)}$ is conformed by the reduced equation

$$\hat{f}_1^{(1)}(z, y) \stackrel{def}{=} y_{zzzz} - z = 0 \quad (4.1)$$

Normal cone is

$$U_1^{(1)} = \{-\mu(1, 5), \mu > 0\} \quad (4.2)$$

Therefor $\omega = -1$, i.e. $z \rightarrow 0$ and $r = 5$. Power solutions are found in the form

$$w = c_5 z^5$$

For c_5 we have

$$c_5 = \frac{1}{120} \quad (4.3)$$

The only power solution is

$$\mathcal{F}_2^{(1)}1 : w = \frac{z^5}{120} \quad (4.4)$$

Compute the critical numbers. The first variation of (2.9) is

$$\frac{\delta \hat{f}_1^{(1)}}{\delta w} = \frac{d^4}{dz^4} \quad (4.5)$$

We get the proper numbers

$$k_1 = 0, \quad k_2 = 1, \quad k_3 = 2, \quad k_4 = 3 \quad (4.6)$$

The cone of the problem

$$\mathcal{K} = \{k > 5\}$$

doesn't consist them. Solution (4.4) is corresponded to two vector indexes $\tilde{Q}_1 = (0, 1)$, $\tilde{Q}_2 = (5, 0)$. Their difference $B = \tilde{Q}_1 - \tilde{Q}_2 = (-5, 1)$ equals to vector $Q_1 - Q_2$. So solution (4.4) is conformed to lattice \mathbf{Z} , which consists of points $Q = (q_1, q_2) = k(-3, 2) + m(-5, 1) = (-3k - 5l, 2k + l)$, where k and l are whole numbers. Points belong to line $q_2 = -1$, if $l = -1 - 2k$. In this case $q_1 = 5 + 7k$. As long as the cone of the problem here is $\mathcal{K} = \{k > 5\}$, the set of the carrier of solution expansion \mathbf{K} takes the form

$$\mathbf{K} = \{5 + 7n, \quad n \in \mathbb{N}\} \quad (4.7)$$

Then the expansion of solution can be written as

$$w(z) = z^5 \left(\frac{1}{120} + \sum_{m=1}^{\infty} c_{5+7m} z^{7m} \right) \quad (4.8)$$

Expansion (4.8) with taking into account three terms takes the form

$$w(z) = \frac{z^5}{120} \left(1 + \frac{13 z^7}{57024} + \frac{2851 z^{14}}{79569008640} + \dots \right) \quad (4.9)$$

Equation (4.1) doesn't have exponential additions and non-power asymptotic forms.

5. Solutions, corresponding to edge $\Gamma_2^{(1)}$.

Edge $\Gamma_2^{(1)}$ is corresponded to the reduced equation

$$\hat{f}_2^{(1)}(z, w) \stackrel{def}{=} w_{zzzz} - 10 w w_{zz} - 5 w^2 + 10 w^3 = 0 \quad (5.1)$$

The normal cone is

$$U_2^{(1)} = \{-\mu(1, -2), \quad \mu > 0\} \quad (5.2)$$

Therefor $\omega = -1$, i.e. $z \rightarrow 0$ and $r = -2$. Hence the solution of equation (5.1) we can find in the form

$$w = c_{-2} z^{-2} \quad (5.3)$$

For c_{-2} we have the determining equation

$$c_{-2}^2 - 8c_{-2} + 12 = 0 \quad (5.4)$$

Consequently we get

$$c_{-2}^{(1)} = 2, \quad c_{-2}^{(2)} = 6 \quad (5.5)$$

The reduced solutions are

$$\mathcal{F}_2^{(1)} 1 : w = 2z^{-2} \quad (5.6)$$

$$\mathcal{F}_2^{(1)} 2 : w = 6z^{-2} \quad (5.7)$$

Let's compute the corresponding critical numbers. The first variation is

$$\frac{\delta f_2^{(1)}}{\delta w} = \frac{d^4}{dz^4} - 10w_{zz} - 10w \frac{d^2}{dz^2} - 10w_z \frac{d}{dz} + 30w^2 \quad (5.8)$$

Applied to solution (5.6), it produces operator

$$\mathcal{L}^{(1)}(z) = \frac{d^4}{dz^4} - \frac{20}{z^2} \frac{d^2}{dz^2} + \frac{40}{z^3} \frac{d}{dz} \quad (5.9)$$

which is corresponded by the characteristic polynomial

$$\nu(k) = k^4 - 6k^3 - 9k^2 + 54k \quad (5.10)$$

Equation

$$\nu(k) = 0 \quad (5.11)$$

has the roots

$$k_1 = -3, \quad k_2 = 0, \quad k_3 = 3, \quad k_4 = 6 \quad (5.12)$$

With reference to solution (5.7) variation (5.8) gives operator

$$\mathcal{L}^{(2)}(z) = \frac{d^4}{dz^4} + \frac{60}{z^2} \frac{d^2}{dz^2} - \frac{120}{z^3} \frac{d}{dz} + \frac{720}{z^4} \quad (5.13)$$

which is corresponded by the characteristic polynomial

$$\nu(k) = k^4 - 6k^3 - 49k^2 + 174k + 720 \quad (5.14)$$

with roots

$$k_1 = -5, \quad k_2 = -3, \quad k_3 = 6, \quad k_4 = 8 \quad (5.15)$$

The cone of the problem here is

$$\mathcal{K} = \{k > -2\} \quad (5.16)$$

Therefor for the reduced solution (5.6) three critical numbers belong to the cone, and there are two critical numbers for the reduced solution (5.7) in the cone of the problem.

The set of the carriers of the solution expansions \mathbf{K} can be written as

$$\mathbf{K} = \{-2 + 7n, \quad n \in \mathbb{N}\} \quad (5.17)$$

Sets $\mathbf{K}(0)$, $\mathbf{K}(0, 3)$ and $\mathbf{K}(0, 3, 6)$ are

$$\begin{aligned} \mathbf{K}(0) &= \{-2 + 7n + 2m, \quad n, m \in \mathbb{N}, \quad n + m \geq 0\} = \\ &= \{-2, 0, 4, 6, 5, 7, 8, \dots\} \end{aligned} \quad (5.18)$$

$$\begin{aligned} \mathbf{K}(0, 3) &= \{-2 + 7n + 2m + 5k, \quad n, m, k \in \mathbb{N}, \quad m + n + k \geq 0\} = \\ &= \{-2, 0, 2, 3, 4, 5, 6, 7, 8, \dots\} \end{aligned} \quad (5.19)$$

$$\begin{aligned} \mathbf{K}(0, 3, 6) &= \{-2 + 7n + 2m + 5k + 8l, \quad n, m, k, l \in \mathbb{N}, \quad m + n + k + l \geq 0\} = \\ &= \{-2, 0, 2, 3, 4, 5, 6, 7, 8, \dots\} \end{aligned} \quad (5.20)$$

In this case the expansion for the solution of equation can be represented as

$$w(z) = \frac{2}{z^2} + \sum_{n,m,k,l} c_{5+7n+2m+2k+8l} z^{5+7n+2m+2k+8l} \quad (5.21)$$

Denote this family as G_2^1 . The critical number 0 doesn't belong to set \mathbf{K} , so the compatibility condition for c_0 holds automatically and c_0 is the arbitrary constant. The critical number 3 also doesn't belong to sets \mathbf{K} and $\mathbf{K}(0)$, therefor the compatibility condition for c_3 holds too and c_3 is the arbitrary constant. But critical number 6 is a member of $\mathbf{K}(0)$ and $\mathbf{K}(0, 6)$, so it is necessary to verify that the compatibility condition for c_6 holds and that c_6 is the arbitrary constant. The calculation shows that in this situation the condition holds and c_6 is the arbitrary constant too. The three-parameter

power expansion of solutions, corresponding to the reduced solution (5.6) takes the form

$$\begin{aligned}
w(z) = & \frac{2}{z^2} + c_0 - \frac{3}{2} c_0^2 z^2 + c_3 z^3 - \frac{5}{2} c_0^3 z^4 + \left(\frac{3}{4} c_0 c_3 - \frac{1}{80} \right) z^5 + c_6 z^6 - \\
& - \frac{1}{280} c_0 z^7 + \left(\frac{153}{352} c_0^5 + \frac{9}{44} c_0 c_6 + \frac{9}{176} c_3^2 \right) z^8 + \left(\frac{19}{12096} c_0^2 - \frac{5}{16} c_0^3 c_3 \right) z^9 + \\
& + \left(\frac{25}{104} c_0^6 - \frac{29}{29120} c_3 - \frac{3}{26} c_0^2 c_6 + \frac{3}{52} c_0 c_3^2 \right) z^{10} + \dots
\end{aligned} \tag{5.22}$$

The carrier of power expansion, corresponding to reduced solution (5.7), is formed by the sets

$$\begin{aligned}
\mathbf{K}(6) = & \{-2 + 7n + 8m, n, m \in \mathbb{N}, m + n \geq 0\} = \\
= & \{-2, 5, 6, 12, 14, 20, 21, 22, 27, 28, 29, 30, 34, 35, 36, 37, 38, 41, \dots\}
\end{aligned} \tag{5.23}$$

$$\begin{aligned}
\mathbf{K}(6, 8) = & \{-2 + 7n + 8m + 10k, n, m, k \in \mathbb{N}, m + n + k \geq 0\} = \\
= & \{-2, 5, 6, 8, 12, 13, 14, 15, 16, 18, 19, 20, 21, \dots\}
\end{aligned} \tag{5.24}$$

The expansion for solution of equation can be written as

$$w(z) = \frac{6}{z^2} + \sum_{n,m,k} c_{5+7n+8m+10k} z^{5+7n+8m+10k} \tag{5.25}$$

Denote this family as G_2^1 . The critical numbers 6 and 8 don't belong to the set \mathbf{K} and the number 8 doesn't belong to the set $\mathbf{K}(6)$. For numbers 6 and 8 the compatibility conditions holds automatically, therefor coefficients c_6 and c_8 are the arbitrary constants. The two-parameter expansion of solution, corresponding to the reduced solution (5.7), is

$$\begin{aligned}
w(z) = & \frac{6}{z^2} + \frac{1}{240} z^5 + c_6 z^6 + c_8 z^8 + \frac{29}{70502400} z^{12} + \\
& + \frac{11}{60480} c_6 z^{13} + \frac{25}{1292} c_6^2 z^{14} + \frac{1}{6804} c_8 z^{15} + \dots
\end{aligned} \tag{5.26}$$

According to [25], the expansions of solutions (5.22) and (5.26) don't have power and exponential additions.

6. Solutions, corresponding to edge $\Gamma_3^{(1)}$.

Edge $\Gamma_3^{(1)}$ is corresponded by the reduced equation

$$\hat{f}_3^{(1)}(z, w) \stackrel{def}{=} 10w^3 - z = 0 \quad (6.1)$$

It has three power solutions

$$\mathcal{F}_3^{(1)}1: \quad w = \varphi^{(1)}(z) = c_{1/3}^{(1)} z^{1/3}, \quad c_{1/3}^{(1)} = \left(\frac{1}{10}\right)^{1/3} \quad (6.2)$$

$$\mathcal{F}_3^{(1)}2: \quad w = \varphi^{(2)}(z) = c_{1/3}^{(2)} z^{1/3}, \quad c_{1/3}^{(2)} = \left(\frac{1}{2} + i\sqrt{3}\right) \left(\frac{1}{10}\right)^{1/3} \quad (6.3)$$

$$\mathcal{F}_3^{(1)}3: \quad w = \varphi^{(3)}(z) = c_{1/3}^{(3)} z^{1/3}, \quad c_{1/3}^{(3)} = \left(\frac{1}{2} - i\sqrt{3}\right) \left(\frac{1}{10}\right)^{1/3} \quad (6.4)$$

The shifted carrier of reduced solutions (6.2) – (6.4) gives a vector

$$B = \left(\frac{1}{3}, -1\right) \quad (6.5)$$

which equals a third of vector $Q_2 - Q_1$. Therefor we explore the lattice, generated by vectors $Q_3 - Q_1$ and B . We have $Q = (q_1, q_2) = k(-5, 1) + l(-3, 2) + m\left(\frac{1}{3}, -1\right) = (2k + \frac{l}{3}, k - l)$, where k, l and m are the whole numbers. At the line $q_2 = -1$ we have $k + 2l - l - m = -1$, wherefrom $m = k + 2l + 1$ and $q_1 = \frac{(1+7l-14k)}{3}$. And so the carrier of solution is

$$\mathbf{K} = \left\{ k = \frac{1-7n}{3}, \quad n \in \mathbb{N} \right\} \quad (6.6)$$

and the expansions of solutions take the form

$$G_3^{(1)}l: \quad w = \varphi^{(l)}(z) = c_{1/3}^{(l)} z^{1/3} + \sum_{n=1}^{\infty} c_{(1-7n)/3}^{(l)} z^{(1-7n)/3} \quad (6.7)$$

Here $c_{1/3}^{(l)}$ can be found from reduced solutions (6.2) – (6.4), coefficients $c_{(1-7n)/3}^{(l)}$ are computed sequentially. The calculating of the coefficient $c_{-2}^{(l)}$

gives the result $c_{-2} = -\frac{1}{18}$. The expansion of solution with taking into account five terms is

$$\begin{aligned} \varphi^{(l)}(z) = & c_{1/3} z^{1/3} - \frac{1}{18} z^{-2} - \frac{7}{108} \frac{1}{c_{1/3}} z^{-13/3} - \\ & - \frac{4199}{17496} c_{1/3}^{-2} z^{-20/3} - \frac{28006583}{23514624} \frac{1}{c_{1/3}^3} z^{-9} + \dots \end{aligned} \quad (6.8)$$

The obtained expansions seem to be divergent ones.

7. Exponential additions of the first level.

Let's find the exponential additions to solutions (6.2)-(6.4). We look for the solutions in the form

$$w = \varphi^{(l)}(z) + u^{(l)}, \quad l = 1, 2, 3$$

The reduced equation for the addition $u^{(l)}$ is

$$M_l^{(1)}(z)u^{(l)} = 0 \quad (7.1)$$

where $M_l^{(1)}(z)$ is the first variation at the solution $w = \varphi^{(l)}(z)$. As long as

$$\frac{\delta f}{\delta w} = \frac{d^4}{dz^4} - 10w_{zz} - 10w \frac{d^2}{dz^2} - 10w_z \frac{d}{dz} + 30w^2 \quad (7.2)$$

then

$$M_l^{(1)}(z) = \frac{d^4}{dz^4} - 10\varphi_{zz}^{(l)} - 10\varphi^{(l)} \frac{d^2}{dz^2} - 10\varphi_z^{(l)} \frac{d}{dz} + 30\varphi^{(l)2} \quad (7.3)$$

Equation (7.1) takes the form

$$\begin{aligned} \frac{d^4 u^{(l)}}{dz^4} - 10\varphi_{zz}^{(l)} u^{(l)} - 10\varphi^{(l)} \frac{d^2 u^{(l)}}{dz^2} - 10\varphi_z^{(l)} \frac{du^{(l)}}{dz} + 30\varphi^{(l)2} u^{(l)} = 0, \\ l = 1, 2, 3 \end{aligned} \quad (7.4)$$

Suppose that

$$\zeta^{(l)} = \frac{d \ln u^{(l)}}{dz} \quad (7.5)$$

then from (7.5) we have

$$\begin{aligned}\frac{du^{(l)}}{dz} &= \zeta^{(l)} u^{(l)}, & \frac{d^2 u^{(l)}}{dz^2} &= \zeta_z^{(l)} u^{(l)} + \zeta^{(l)2} u^{(l)} \\ \frac{d^3 u^{(l)}}{dz^3} &= \zeta_{zz}^{(l)} u^{(l)} + 3\zeta^{(l)} \zeta_z^{(l)} u^{(l)} + \zeta^{(l)3} u^{(l)} \\ \frac{d^4 u^{(l)}}{dz^4} &= \zeta_{zzz}^{(l)} u^{(l)} + 4\zeta^{(l)} \zeta_{zz}^{(l)} u^{(l)} + 3\zeta_z^{(l)2} u^{(l)} + 6\zeta^{(l)2} \zeta_z^{(l)} u^{(l)} + \zeta^{(l)4} u^{(l)}\end{aligned}$$

By substituting the derivatives

$$\frac{du^{(l)}}{dz}, \quad \frac{d^2 u^{(l)}}{dz^2}, \quad \frac{d^4 u^{(l)}}{dz^4}$$

into the equation (7.4) we get the reduced equation in the form

$$\begin{aligned}u^{(l)} \left[\zeta_{zzz}^{(l)} + 4\zeta^{(l)} \zeta_{zz}^{(l)} + 3\zeta_z^{(l)2} + 6\zeta^{(l)2} \zeta_z^{(l)} + \right. \\ \left. + \zeta^{(l)4} - 10\varphi_{zz}^{(l)} - 10\varphi^{(l)} \zeta_z^{(l)} - 10\varphi^{(l)} \zeta^{(l)2} - 10\varphi_z^{(l)} \zeta^{(l)} + 30\varphi^{(l)2} \right] = 0\end{aligned} \quad (7.6)$$

Let's find the power expansions for solutions of equation (7.6). The carrier of equation (7.6) consists of points

$$\begin{aligned}Q_1 &= (-3, 1), \quad Q_2 = (-2, 2), \quad Q_3 = (-1, 3), \\ Q_4 &= (0, 4), \quad Q_5 = \left(\frac{1}{3}, 2\right), \quad Q_6 = \left(\frac{2}{3}, 0\right), \quad Q_7 = \left(-\frac{2}{3}, 1\right), \\ Q_8 &= \left(-\frac{5}{3}, 0\right), \quad Q_{8+n} = \left(-\frac{5+7n}{3}, 0\right), \quad Q_{9+m} = \left(-\frac{2+7m}{3}, 1\right), \\ Q_{10+k} &= \left(\frac{1-7k}{3}, 2\right), \quad Q_{11+l} = \left(\frac{12-14l}{3}, 0\right), \quad m, n, k, l \in \mathbb{N}\end{aligned} \quad (7.7)$$

The closing of convex hull of points of the carrier of equation (7.6) is the strip. It is represented at fig. 3.

The periphery of the strip contains edges $\Gamma_j^{(1)}$ ($j = 1, 2, 3$) with normal vectors $N_1 = (6, 1)$, $N_2 = (0, -1)$, $N_3 = (0, 1)$. It should take up edge $\Gamma_1^{(1)}$ only. This edge is corresponded by the reduced equation

$$h_1^{(1)}(z, \zeta) \stackrel{def}{=} \zeta^4 - 10\varphi^{(l)} \zeta^2 + 30\varphi^{(l)2} = 0 \quad (7.8)$$

Wherefrom we have

$$\zeta^2 = \left(5 + (-1)^{m-1} i \sqrt{5}\right) \varphi^{(l)}, \quad m = 1, 2 \quad (7.9)$$

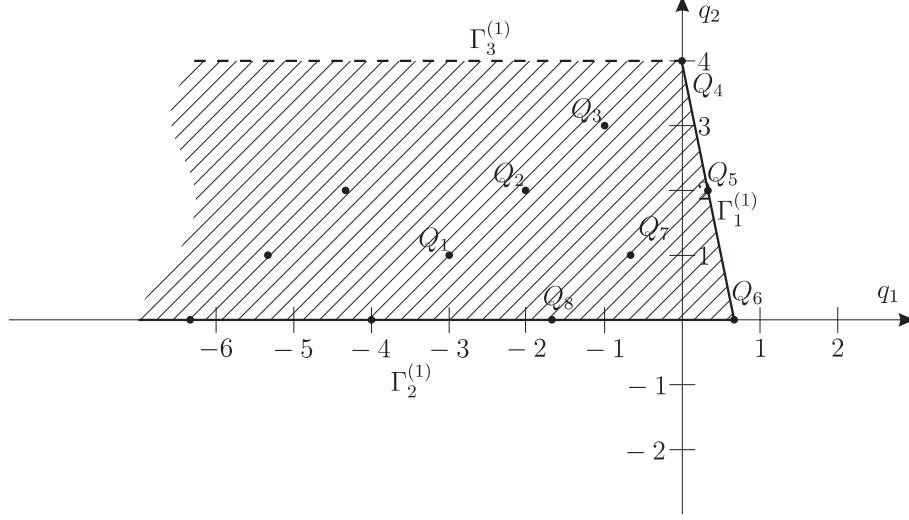


Figure 3:

We obtain twelve solutions of equation (7.8)

$$\zeta^{(l,m,k)} = g_{1/6}^{(l,m,k)} z^{1/6}, \quad l = 1, 2, 3; \quad m, k = 1, 2 \quad (7.10)$$

where

$$g_{1/6}^{(1,m,k)} = \left(\frac{1}{10} \right)^{1/3} (-1)^{k-1} \left(5 + (-1)^{m-1} i \sqrt{5} \right)^{1/2}, \quad (7.11)$$

$$m, k = 1, 2$$

$$g_{1/6}^{(2,m,k)} = \left(\frac{1}{2} + i \sqrt{3} \right) \left(\frac{1}{10} \right)^{1/3} (-1)^{k-1} \left(5 + (-1)^{m-1} i \sqrt{5} \right)^{1/2}, \quad (7.12)$$

$$m, k = 1, 2$$

$$g_{1/6}^{(3,m,k)} = \left(\frac{1}{2} - i \sqrt{3} \right) \left(\frac{1}{10} \right)^{1/3} (-1)^{k-1} \left(5 + (-1)^{m-1} i \sqrt{5} \right)^{1/2}, \quad (7.13)$$

$$m, k = 1, 2$$

The reduced equation is algebraic one, so it have no critical numbers. Let's compute the carrier of the expansion for solution of equation (7.6). The shifted carrier of equation (7.6) is contained in a lattice, generated by vectors $B_1 = (\frac{7}{3}, 0)$, $B_2 = (1, 1)$. The shifted carrier of solutions (7.10) gives

rise to vector $B_3 = (-\frac{1}{6}, 1)$. The difference $B_2 - B_3 = (\frac{7}{6}, 0) = \frac{1}{2}B_2 \stackrel{def}{=} B_4$. Therefor, vectors B_1, B_2 and B_3 generate the same lattice as vectors B_2, B_4 . Points of this lattice can be written as

$$Q = (q_1, q_2) = k(1, 1) + m\left(\frac{7}{6}, 0\right) = \left(k + \frac{7m}{6}, k\right)$$

At the line $q_2 = -1$ we have $k = -1$, and so $q_1 = -1 + \frac{7m}{6}$. As long as the cone of the problem here is $\mathcal{K} = \{k < \frac{1}{6}\}$, then the set of the carriers of expansions \mathbf{K} is

$$\mathbf{K} = \left\{ \frac{1-7n}{6}, n \in \mathbb{N} \right\} \quad (7.14)$$

The expansion for solution of equation (7.6) takes the form

$$\zeta^{(l,m,k)} = g_{1/6}^{(l,m,k)} z^{1/6} + \sum_n g_{(1-7n)/6}^{(l,m,k)} z^{(1-7n)/6}, \quad (7.15)$$

$$l = 1, 2, 3; \quad m = 1, 2; \quad k = 1, 2$$

Coefficients $g_{1/6}^{(l,m,k)}$ are determined by expressions (7.11), (7.12) and (7.13). Coefficient $g_{-1}^{(l,m,k)}$ takes on a value

$$g_{-1}^{(l,m,k)} = -\frac{1}{4} \quad (7.16)$$

The expansion of solution with taking into account four terms takes the form

$$\begin{aligned} \zeta^{(l,m,k)} = & g_{1/6} z^{1/6} - \frac{1}{4} z^{-1} - \frac{7}{288} \frac{(30 g_{1/6}^2 - 7 \cdot 10^{2/3})}{g_{1/6} (2 g_{1/6}^2 - 10^{2/3})} z^{-13/6} - \\ & - \frac{49}{1728} \frac{30 g_{1/6}^4 - 6 \cdot 10^{2/3} g_{1/6}^2 + 35 \sqrt[3]{10}}{g_{1/6}^2 (5 \sqrt[3]{10} - 2 \cdot 10^{2/3} g_{1/6}^2 + 2 g_{1/6}^4)} z^{-10/3} + \dots \end{aligned} \quad (7.17)$$

In view of (7.5) we can find the additions $u^{(l,m,k)}(z)$. We have

$$u^{(l,m,k)}(z) = C \exp \int \zeta^{(l,m,k)}(z) dz$$

Wherefrom we get

$$u^{(l,m,k)}(z) = C_1 z^{-1/4} \exp \left[\frac{6}{7} g_{1/6}^{(l,m,k)} z^{7/6} + \sum_{n=2}^{\infty} \frac{6}{7(1-n)} g_{(1-7n)/6}^{(l,m,k)} z^{7(1-n)/6} \right]$$

$$l = 1, 2, 3; \quad m = 1, 2; \quad k = 1, 2$$

(7.18)

Here C_1 and farther C_2 and C_3 are the arbitrary constants. Addition $u^{(l,m,k)}(z)$ near $z \rightarrow \infty$ is the exponentially small one in those sectors of complex plane z , where

$$\operatorname{Re} \left[g_{1/6}^{(l,m,k)} z^{1/6} \right] < 0 \quad (7.19)$$

Thus for three expansions $G_3^{(1)}l$ we get four one-parameter family of additions $G_3^{(1)}lG_1^1mk$, where $m = 1, 2$ and $k = 1, 2$.

8. Exponential additions of the second level.

Let's find exponential additions of the second level $v^{(p)}$, i.e. the additions to solutions $u^{(l,m,k)}(z)$. The reduced equation for addition $v^{(p)}$ is

$$M_p^{(2)}(z)v^{(p)} = 0 \quad (8.1)$$

where operator $M_p^{(2)}$ is the first variation of (7.6). Equation (8.5) for $v = v^{(p)}$ takes the form

$$\begin{aligned} & \frac{d^3 v}{dz^3} + 4\zeta_{zz}v + 4\zeta v_{zz} + 6\zeta_z v_z + 12\zeta\zeta_z v + \\ & + 6\zeta^2 v_z + 4\zeta^3 v - 10\varphi^{(l)}v_z - 20\varphi^{(l)}\zeta v - 10\varphi_z^{(l)}v = 0 \end{aligned} \quad (8.2)$$

Assumed that

$$\frac{d \ln v}{dz} = \xi \quad (8.3)$$

we have

$$\frac{dv}{dz} = \xi v, \quad \frac{d^2 v}{dz^2} = \xi_z v + \xi^2 v, \quad \frac{d^3 v}{dz^3} = \xi_{zz}v + 3\xi\xi_z v + \xi^3 v \quad (8.4)$$

From (8.2) we get equation

$$\begin{aligned} & \xi_{zz} + 3\xi\xi_z + \xi^3 + 4\zeta_{zz} + 4\zeta_z\xi + 4\xi^2\zeta + 6\xi\zeta_z + 12\zeta\zeta_z + 6\xi\zeta^2 + \\ & + 4\zeta^3 - 10\varphi^{(l)}\xi - 20\zeta\varphi^{(l)} - 10\varphi_z^{(l)} = 0 \end{aligned} \quad (8.5)$$

Monomials of equation (8.5) is corresponded by the points

$$\begin{aligned}
M_1 &= (-2, 1), \quad M_2 = (-1, 2), \quad M_3 = (0, 3), \quad M_4 = \left(-\frac{11}{6}, 0\right), \\
M_5 &= \left(-\frac{5}{6}, 1\right), \quad M_6 = \left(\frac{1}{6}, 2\right), \quad M_7 = \left(-\frac{5}{6}, 1\right), \\
M_8 &= \left(-\frac{2}{3}, 0\right), \quad M_9 = \left(\frac{1}{3}, 1\right), \quad M_{10} = \left(\frac{1}{2}, 0\right), \\
M_{11} &= \left(\frac{1}{3}, 1\right), \quad M_{12} = \left(\frac{1}{2}, 0\right), \quad M_{13} = \left(-\frac{2}{3}, 0\right), \quad \dots
\end{aligned} \tag{8.6}$$

The carrier of the equation (8.5) is determined by points of the set (8.6). The convex set forms the strip, which is similar to the strip, represented at fig. 3. It should examine edge $\Gamma_1^{(1)}$, which is passing through points

$$Q_1 = \left(\frac{1}{2}, 0\right), \quad Q_2 = \left(\frac{1}{3}, 1\right), \quad Q_3 = (0, 3) \tag{8.7}$$

The reduced equation, corresponding to this edge, is

$$\xi^3 + 4\xi^2\zeta + 6\xi\zeta^2 + 4\zeta^3 - 20\zeta\varphi^{(l)} - 10\xi\varphi^{(l)} = 0 \tag{8.8}$$

The basis of the lattice, corresponding to the carrier of equation (8.5) is

$$B_1 = (1, 1), \quad B_2 = \left(\frac{7}{6}, 0\right)$$

The solution of equation (8.8) takes the form

$$\xi^{(l,m,k,p)} = r_{1/6}^{(l,m,k,p)} z^{1/6}, \quad m, k = 1, 2; \quad l = 1, 2, 3; \quad p = 1, 2, 3 \tag{8.9}$$

where $r = r_{1/6}^{(l,m,k,p)}$, $p = 1, 2, 3$ are the roots of the equation

$$\begin{aligned}
5r^3 + 4r^2 g_{1/6}^{(l,m,k)} + \left(6g_{1/6}^{(l,m,k)^2} - 10c_{1/3}^{(l)}\right)r + 4g_{1/6}^{(l,m,k)^3} \\
- 20g_{1/6}^{(l,m,k)} c_{1/3}^{(l)} = 0
\end{aligned} \tag{8.10}$$

Equation (8.10) has the roots

$$\begin{aligned}
r_{1/6}^{(l,m,k,1)} &= -2g_{1/6}^{(l,m,k)}, \quad r_{1/6}^{(l,m,k,2)} = -g_{1/6}^{(l,m,k)} + \left(10c_{1/3}^{(l)} - g_{1/6}^{(l,m,k)}\right)^{1/2} \\
r_{1/6}^{(l,m,k,3)} &= -g_{1/6}^{(l,m,k)} - \left(10c_{1/3}^{(l)} - g_{1/6}^{(l,m,k)}\right)^{1/2}
\end{aligned} \tag{8.11}$$

The set of carriers of expansions for solution \mathbf{K} coincides with (7.14). The expansion of solution for $\xi^{(l,m,k,p)}$ takes the form

$$\xi^{(l,m,k,p)} = r_{1/6}^{(l,m,k,p)} z^{1/6} + \sum_{n=1}^{\infty} r_{(1-7n)/6}^{(l,m,k,p)} z^{(1-7n)/6}, \quad (8.12)$$

$$l = 1, 2, 3; \quad m = 1, 2; \quad k = 1, 2; \quad p = 1, 2, 3$$

The computing the coefficient $r_{-1}^{(l,m,k,p)}$ gives a result $r_{-1}^{(l,m,k,p)} = 1/6$. The expansion of solution with taking into account three terms is

$$\begin{aligned} \xi^{(l,m,k,p)} = & r_{1/6} z^{1/6} + \frac{1}{6} z^{-1} + (-30 g_{1/6}^4 - 9 \cdot 10^{2/3} g_{1/6}^2 + \\ & + 150 g_{1/6}^2 c_{1/3} - 35 c_{1/3} 10^{2/3} - 30 r_{1/6}^2 g_{1/6}^2 + 7 r_{1/6}^2 10^{2/3} - \\ & - 60 g_{1/6}^3 r_{1/6} + 6 r_{1/6} g_{1/6} 10^{2/3}) (10^{2/3} - 2 g_{1/6}^2)^{-1} \\ & (6 g_{1/6}^2 - 10 c_{1/3} + 3 r_{1/6}^2 + 8 g_{1/6} r_{1/6})^{-1} g_{1/6}^{-1} z^{-\frac{13}{6}} \end{aligned} \quad (8.13)$$

The exponential addition $v^{(l,m,k,p)}(z)$ to solutions $u^{(l,m,k)}(z)$ is

$$v^{(l,m,k,p)}(z) = C_2 z^{1/6} \exp \left[\frac{6}{7} r_{1/6}^{(l,m,k,p)} z^{7/6} + \sum_{n=2}^{\infty} \frac{6}{7(1-n)} r_{(1-7n)/6}^{(l,m,k,p)} z^{7(1-n)/6} \right],$$

$$l = 1, 2, 3; \quad m = 1, 2; \quad k = 1, 2; \quad p = 1, 2, 3 \quad (8.14)$$

Solutions $v^{(l,m,k,p)}(z)$ seem to be divergent ones too.

9. Exponential additions of the third level.

Let's compute the exponential additions of the third level $y^{(s)}$, i.e. the additions to the solutions $v^{(l,m,k,p)}(z)$. The reduced equation for addition $y^{(s)}$ is

$$M_s^{(3)}(z) y^{(s)} = 0 \quad (9.1)$$

Operator $M_s^{(3)}$ is the first variation of (8.5). Equation (9.1) for $y = y^{(l,m,k,p,s)}$ takes the form

$$\begin{aligned} y_{zz} + 3\xi_z y + 3\xi y_z + 3\xi^2 y + 4\zeta y_z + 8\xi\zeta y + \\ + 6\zeta_z y + 6\zeta^2 y - 10\varphi^{(l)} y = 0 \end{aligned} \quad (9.2)$$

Using the substitute

$$\frac{d \ln y}{dz} = \eta \quad (9.3)$$

we obtain

$$\frac{dy}{dz} = \eta y, \quad \frac{d^2 y}{dz^2} = \eta_z y + \eta^2 y \quad (9.4)$$

From (9.4) we have equation

$$\eta_z + \eta^2 + 3\xi_z + 3\xi\eta + 3\xi^2 + 4\eta\zeta + 8\xi\zeta + 6\zeta_z + 6\zeta^2 - 10\varphi^{(l)} = 0 \quad (9.5)$$

Monomials of equation (9.5) is corresponded by points

$$\begin{aligned} M_1 = (-1, 1), \quad M_2 = (0, 2), \quad M_3 = \left(-\frac{5}{6}, 0\right), \quad M_4 = \left(\frac{1}{6}, 1\right), \\ M_5 = \left(\frac{1}{3}, 0\right), \quad M_6 = \left(\frac{1}{6}, 0\right), \quad M_7 = \left(\frac{1}{3}, 0\right), \quad M_8 = \left(-\frac{5}{6}, 0\right), \\ M_9 = \left(\frac{1}{3}, 0\right), \quad M_{10} = \left(\frac{1}{3}, 0\right), \quad \dots \end{aligned} \quad (9.6)$$

The carrier of equation (9.5) is formed by points (9.6). The convex set forms the strip, which is similar to the strip, represented at fig. 3. It should examine edge $\Gamma_1^{(1)}$, which is passing through points

$$Q_1 = \left(\frac{1}{3}, 0\right), \quad Q_2 = \left(\frac{1}{6}, 1\right), \quad Q_3 = (0, 2) \quad (9.7)$$

The reduced equation, corresponding to this edge, is

$$\eta^2 + 3\xi\eta + 4\eta\zeta + 8\xi\zeta + 6\zeta^2 - 10\varphi^{(i)} = 0 \quad (9.8)$$

The basis of the lattice, corresponding to the carrier of equation (9.7), is

$$B_1 = (1, 1), \quad B_2 = \left(\frac{7}{6}, 0\right)$$

The solutions of equation (9.8) takes the form

$$\begin{aligned} \eta^{(l,m,k,p,s)} = q^{(l,m,k,p,s)} z^{1/6} \\ l = 1, 2, 3; \quad m, k = 1, 2; \quad p = 1, 2, 3; \quad s = 1, 2; \end{aligned} \quad (9.9)$$

where $q^{(l,m,k,p,s)} = q$ are the roots of equation

$$q^2 + 3q r_{1/6}^{(l,m,k,p)} + 4q g_{1/6}^{(l,m,k)} + 8r_{1/6}^{(l,m,k,p)} g_{1/6}^{(l,m,k)} + 6g_{1/6}^{(l,m,k)^2} - 10c_{1/3}^{(l)} = 0 \quad (9.10)$$

The roots of equation (9.10) are

$$q_{1/6}^{(l,m,k,p,s)} = -\frac{3}{2} r_{1/6}^{(l,m,k,p)} - 2g_{1/6}^{(l,m,k)} + (-1)^{s-1} \left(\frac{9}{4} r_{1/6}^{(l,m,k,p)^2} - 2r_{1/6}^{(l,m,k,p)} g_{1/6}^{(l,m,k)} - 2g_{1/6}^{(l,m,k)^2} + 10c_{1/3}^{(l)} \right)^{(1/2)},$$

$$l = 1, 2, 3; \quad m, k = 1, 2; \quad p = 1, 2, 3; \quad s = 1, 2; \quad (9.11)$$

The set of carriers of expansions for solution **K** coincides with (7.14). The expansion of solution for $\eta^{(l,m,k,p,s)}$ takes the form

$$\eta^{(l,m,k,p,s)} = q_{1/6}^{(l,m,k,p,s)} z^{1/6} + \sum_{n=1}^{\infty} q_{(1-7n)/6}^{(l,m,k,p,s)} z^{(1-7n)/6}, \quad (9.12)$$

$$l = 1, 2, 3; \quad m = 1, 2; \quad k = 1, 2; \quad p = 1, 2, 3; \quad s = 1, 2;$$

Coefficients $q_{1/6}^{(l,m,k,p,s)}$, $s = 1, 2$ are determined by formulas (9.11). The computing of the coefficient $q_{-1}^{(l,m,k,p,s)}$ gives a result $q_{-1}^{(l,m,k,p,s)} = 1/6$. Exponential addition $y^{(s,p,l,m,k)}(z)$ to the solutions $v^{(l,m,k,p)}(z)$ is

$$y^{(l,m,k,p,s)}(z) = C_3 z^{1/6} \exp \left[\frac{6}{7} q_{1/6}^{(l,m,k,p,s)} z^{7/6} + \sum_{n=2}^{\infty} \frac{6}{7(1-n)} q_{(1-7n)/6}^{(l,m,k,p,s)} z^{7(1-n)/6} \right] \quad (9.13)$$

$$l = 1, 2, 3; \quad m = 1, 2; \quad k = 1, 2; \quad p = 1, 2, 3; \quad s = 1, 2$$

Thus we find three levels of the exponential additions to the expansions for solutions of equation near point $z = \infty$. Solution $w(z)$ at $z \rightarrow \infty$ with taking into account the exponential additions has the expansion

$$w(z) = c_{1/3}^{(l)} z^{1/3} - \frac{1}{18z^2} + \sum_{n=2}^{\infty} c_{(1-7n)/3}^{(l)} z^{(1-7n)/3} + C_1 z^{-1/4} \exp\{F_1(z) + C_2 z^{1/6} \exp\{F_2(z) + C_3 z^{1/6} \exp\{F_3(z)\}\}\} \quad (9.14)$$

where $c_{1/3}^{(l)}$ can be computed by formulas (6.2), (6.3) and (6.4); $F_1(z) = F_1^{(l,m,k)}(z)$, $F_2(z) = F_2^{(l,m,k,p)}(z)$ and $F_3(z) = F_3^{(l,m,k,p,s)}(z)$, ($l = 1, 2, 3$; $m, k = 1, 2$; $p = 1, 2, 3$; $s = 1, 2$) are

$$F_1^{(l,m,k)}(z) = \frac{6}{7} g_{1/6}^{(l,m,k)} z^{7/6} + \sum_{n=2}^{\infty} \frac{6}{7(1-n)} g_{(1-7n)/6}^{(l,m,k)} z^{7(1-n)/6} \quad (9.15)$$

$$F_2^{(p,l,m,k)}(z) = \frac{6}{7} r_{1/6}^{(l,m,k,p)} z^{7/6} + \sum_{n=2}^{\infty} \frac{6}{7(1-n)} r_{(1-7n)/6}^{(l,m,k,p)} z^{7(1-n)/6} \quad (9.16)$$

$$F_3^{(l,m,k,p,s)}(z) = \frac{6}{7} q_{1/6}^{(l,m,k,p,s)} z^{7/6} + \sum_{n=2}^{\infty} \frac{6}{7(1-n)} q_{(1-7n)/6}^{(l,m,k,p,s)} z^{7(1-n)/6} \quad (9.17)$$

Coefficients $g_{1/6}^{(l,m,k)}$, $r_{1/6}^{(l,m,k,p)}$ and $q_{1/6}^{(l,m,k,p,s)}$ are defined by formulas (7.11), (7.12), (7.13), (8.11) and (9.11). The other coefficients are computed sequentially.

10. Summary of the results and discussion.

For the solutions of fourth-order analog to the first Painlevé equation (2.1) it is obtained the following expansions.

About a point $z = 0$:

1. Four-parameter (with arbitrary constants c_0 , c_1 , c_2 and c_3) family $G_1^{(0)}1$ of expansion for solution (3.11).
 2. Three-parameter (with arbitrary constants c_1 , c_2 and c_3) family $G_1^{(0)}2$ of expansion (3.12).
 3. Two-parameter (with arbitrary constants c_2 and c_3) family $G_1^{(0)}3$ of expansion (3.13).
 4. One-parameter (with arbitrary constant c_3) family $G_1^{(0)}4$ of expansion (3.15).
- Families $G_1^{(0)}2$, $G_1^{(0)}3$ and $G_1^{(0)}4$ are the special cases of family $G_1^{(0)}1$; $G_1^{(0)}3$ and $G_1^{(0)}4$ are the special cases of $G_1^{(0)}2$; $G_1^{(0)}4$ is the special case of $G_1^{(0)}3$.
5. Family $G_1^{(1)}1$ of expansion (4.8) of solution, which is the special case of families $G_1^{(0)}1$, $G_1^{(0)}2$, $G_1^{(0)}3$ and $G_1^{(0)}4$.
 6. Three-parameter (with arbitrary constants c_0 , c_3 and c_6) family $G_2^{(1)}1$ of expansion (5.21) for solution of equation (2.1).

7. Two-parameter (with arbitrary constants c_6 and c_8) family $G_2^{(1)}2$ of expansion (5.25) for solution of equation (2.1).

All listed expansions converge for sufficiently small $|z|$.

About a point $z = \infty$:

8. Three expansions $G_3^{(1)}l$ ($l = 1, 2, 3$), described by formulas (6.2), (6.3) and (6.4). For each of these expansions it is found four exponential additions $G_3^{(1)}lG_1^1mk$ ($m, k = 1, 2$) expressed by formula (7.18). For them it is also computed exponential additions $G_3^{(1)}lG_1^1mkG_1^{(1)}p$ ($m, k = 1, 2; p = 1, 2, 3$), and then the proper differential additions $G_3^{(1)}lG_1^{(1)}mkG_1^{(1)}pG_1^{(1)}s$ ($m, k = 1, 2; p = 1, 2, 3; s = 1, 2$) are found too.

The existence and analyticity of expansions, described in items 1-7, follow from Cauchy theorem. Families $G_2^{(1)}l$ and $G_2^{(1)}2$ were first found in the paper [13]. However the structure of expansions $G_2^{(1)}1$ and $G_2^{(2)}2$ was not discussed earlier. The other families of expansions of solution are found for the first time.

Comparing the power expansions of equation (2.1) with power expansions of Painlevé equations $P_1 \div P_6$ [?, ?, 26–34] we note, that they differ. This fact can be interpreted as the additional case for the hypothesis, that the fourth-order equation (2.1) determines new transcendental functions just as equations $P_1 \div P_6$.

11. Appendix. The computation of the basis of the plane lattice.

Let there is a set \mathbf{S} of points Q_1, \dots, Q_m on the plane \mathbb{R}^2 , and there is a zero among them. Our aim is to compute the basis B_1, B_2 of the minimal lattice \mathbf{Z} , which contains all the points of set \mathbf{S} . The minimality of lattice \mathbf{Z} means that there is no other lattice $\mathbf{Z} \subset \mathbf{Z}$ and $\mathbf{Z}_1 \neq \mathbf{Z}$, which also contains set \mathbf{S} . The computation is divided into three steps.

Step 1. Let $Q_m = 0$, and the others $Q_j \neq 0$. For all pairs of vectors Q_j, Q_k , $1 \leq j, k < m, j \neq k$ compose the determinants

$$\det(Q_j Q_k) \stackrel{def}{=} \Delta_{jk}. \quad (11.1)$$

Among pairs with $\Delta_{jk} \neq 0$ we find one with $|\Delta_{jk}| = \min |\Delta_{jk}| \neq 0$ in all $j, k = 1, \dots, m-1$. If there are few such pairs, we can take any of them. Suppose for the sake of simplicity that it is pair Q_1, Q_2 . Other pairs Q_3, \dots, Q_{m-1} are arbitrary ordered.

Step 2. Let's find the basis of the lattice, generated by vectors Q_1, Q_2, Q_3 . Let $Q_3 = aQ_1 + bQ_2$, where a and b are the rational quantities. Denote integer part of number $a \in \mathbb{R}$ as $[a]$ and the fractional part as $\{a\}$, i.e. $\{a\} = a - [a]$. Denote $Q'_3 = \{a\}Q_1 + \{b\}Q_2$. Suppose that $\min |\det(Q'_3 Q_i)|$ for $i = 1, 2$ reaches at $i = 1$. Then we take Q_1 and Q'_3 as the basis vectors and use them to express Q_2 , i.e. we get $Q_2 = a_1 Q_1 + b Q'_3$. Replace vector Q_2 by $Q'_2 = \{a_1\}Q_1 + \{b\}Q'_3$. Among three vectors Q_1, Q'_2, Q'_3 we find the pair with the least modulus of determinant. Using this pair we distribute the third vector, take his fractional part and so on. At some step l we obtain that the fractional part of the third vector equals zero. The latest pair of vectors $Q_2^{(l)}, Q_3^{(l)}$ gives the basis of minimal lattice, containing the points Q_1, Q_2, Q_3 .

Step 3. For vectors $Q_2^{(l)}, Q_3^{(l)}, Q_4$ we realize step 2 and get vectors \tilde{Q}_3, \tilde{Q}_4 and so on. After looking through all $Q_j, j \leq m-1$, we get the pair of vectors Q_{m-2}^*, Q_{m-1}^* , which is the basis of minimal lattice, containing the set \mathbf{S} .

Remark. The analogous algorithm allows as to find the basis of minimal lattice in \mathbb{R}^n , containing the given finite set \mathbf{S} . If $n = 1$ it's the Euclid algorithm.

Example 8.1. Let's consider equation (1.4). It's carrier consists of six points (2.1). Move them by vector $Q_4 = (1, 0)$. We obtain

$$Q'_1 = (-5, 1), \quad Q'_2 = (-3, 2), \quad Q'_3 = (-1, 3), \quad Q'_4 = 0$$

For vectors Q'_1, Q'_2, Q'_3 we compute the pairwise determinants

$$\Delta_{12} = \begin{vmatrix} -5 & 1 \\ -3 & 2 \end{vmatrix} = -7, \quad \Delta_{13} = \begin{vmatrix} -5 & 1 \\ -1 & 3 \end{vmatrix} = -14, \quad \Delta_{23} = \begin{vmatrix} -3 & 2 \\ -1 & 3 \end{vmatrix} = -7. \quad (11.2)$$

Thus as the initial pair we can use vectors Q'_1, Q'_2 or Q'_2, Q'_3 . Let's take Q'_1, Q'_2 to fix the idea. We look for the expansion $Q'_3 = aQ'_1 + bQ'_2 = a(-5, 1) + b(-3, 2)$, for that we are to solve the linear system of equations

$$\begin{aligned} -5a - 3b &= -1, \\ a + 2b &= 3. \end{aligned} \quad (11.3)$$

We get $a = -1, b = 2$. As long as $\{a\} = \{b\} = 0$, then the vectors $B_1 = Q'_1$ and $B_2 = Q'_2$ generate the basis of the lattice of shifted carrier of equation (1.4).

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